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* Complete Metric Space

It is a metric space in which every Cauchy sequence is convergent.

* The real line is a complete metric space.

* The complex plane is also complete.

Its Proof Let $\{z_n\}$ be a cauchy sequence of complex numbers. Then $z_n = a_n + ib_n$.
~~be a Cauchy sequence of complex numbers.~~

$\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers

$\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ both are themselves Cauchy sequences of real numbers

because $|a_m - a_n| \leq |z_m - z_n|$ and

$|b_m - b_n| \leq |z_m - z_n|$.

since real line is complete metric space.

$\Rightarrow \exists$ real numbers a and b such that

$a_n \to a$ and $b_n \to b$.

Now, $|z_n - z| = |(a_n + ib_n) - (a+ib)|$

$= |(a_n - a) + i(b_n - b)|$

$\leq |a_n - a| + |b_n - b|$

Take $\lim_{n\to\infty}$ both sides and using the fact

that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$

because real line is a complete metric space
$\Rightarrow$ real ~~spac~~ sequences are cgt.

$\therefore \lim_{n\to\infty} |z_n - z| \leq \lim_{n\to\infty} |a_n - a| + \lim_{n\to\infty} |b_n - b|$

$\leq 0 + 0$

$\Rightarrow \lim_{n\to\infty} z_n = z \Rightarrow$ complex plane is convergent

$\Rightarrow$ complex plane is complete metric space.

Another Examples of complete metric spaces

$\boxed{\text{I}}$ (X, d) be the metric space with

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

$\boxed{\text{II}}$ $d(x, y) = |x - y|$ i.e. the usual metric on $\mathbb{R}$.